# MATH 110 – SOLUTIONS TO THE SECOND PRACTICE MIDTERM

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(1) (a) Write out the definition of isomorphic vector spaces

**Definition:** Two vector spaces $V$ and $W$ are said to be **isomorphic** if there exists a linear transformation $T : V \longrightarrow W$ which is both injective and surjective.

– *Injective* means that whenever $T(v) = 0$ for $v \in V$, then $v = 0$.

– *Surjective* means that for all $w \in W$, there exists $v \in V$ such that $T(v) = w$

(b) Prove that if the dimension of two finite-dimensional vector spaces is equal, then the two vector spaces are isomorphic

Suppose $V$ and $W$ are such that $dim(V) = dim(W) = n < \infty$.

Let $(v_1, \cdots, v_n)$ be a basis for $V$ and $(w_1, \cdots, w_n)$ be a basis for $W$.

Define $T : V \to W$ to be the unique linear transformation with the property that $T(v_i) = w_i$. The existence of such a linear transformation is guaranteed by the **linear extension lemma** (exercise 3 in Homework 6) [1].

We **claim** that this $T$ gives us the desired isomorphism. For this, the only things we need to check is that $T$ is injective and $T$ is surjective.

$T$ is injective:
Suppose $T(v) = 0$ for $v \in V$.

Then, since $(v_1, \cdots, v_n)$ is a basis for $V$, it spans $V$, hence there exist constants $a_1, \cdots, a_n$ such that $v = a_1 v_1 + \cdots + a_n v_n$

But then:

$$\begin{aligned} T(v) = 0 \Rightarrow\ & T(a_1 v_1 + \cdots + a_n v_n) = 0 \\ \Rightarrow\ & a_1 T(v_1) + \cdots + a_n T(v_n) = 0 \\ \Rightarrow\ & a_1 w_1 + \cdots + a_n w_n = 0 \end{aligned}$$

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[1] Write this!!!

Where the third implication followed from the fact that $T(v_i) = w_i$ for $i = 1, \cdots, n$.

But since $(w_1, \cdots, w_n)$ is a basis for $W$, it is linearly independent, and hence $a_1 = \cdots = a_n = 0$.

But then:

$$v = a_1 v_1 + \cdots + a_n v_n = 0v_1 + \cdots + 0v_n = 0$$

So $v = 0$, and hence $T$ is injective.

T is surjective Let $w$ be an arbitrary vector in $W$.

Then, since $(w_1, \cdots, w_n)$ is a basis for $W$, it spans $W$, and hence there exist constants $a_1, \cdots, a_n$ such that $w = a_1 w_1 + \cdots + a_n w_n$.

*Define* $v = a_1 v_1 + \cdots + a_n v_n$.

Then:

$$\begin{aligned} T(v) =& T(a_1 v_1 + \cdots + a_n v_n) \\ =& a_1 T(v_1) + \cdots + a_n T(v_n) \\ =& a_1 w_1 + \cdots + a_n w_n = w \end{aligned}$$

Where the third equality followed from the fact that $T(v_i) = w_i$ for $i = 1, \cdots, n$.

So we found $v$ such that $T(v) = w$, and hence $T$ is surjective.

Therefore, since $T$ is an injective and surjective linear transformation, and therefore by definition, $V$ is isomorphic to $W$. □

(2) (a) Write out the definition of an invariant subspace.

**Definition:** If $T : V \longrightarrow V$ is a linear transformation and $U$ is a subspace of $V$, then $U$ is said to be **invariant under** $T$ if whenever, $u$ is in $U$, then $T(u)$ is in $U$.

(b) Suppose that $T \in \mathcal{L}(V)$. Prove that if $U_1, \cdots, U_m$ are subspaces of $V$ invariant under $T$, then $U_1 + \cdots + U_m$ is also invariant under $T$.

Suppose $u$ is in $U_1 + \cdots + U_m$.

Then, by definition of $U_1 + \cdots + U_m$, $u = u_1 + \cdots + u_m$, where each $u_i$ is in $U_i$.

Then:

$$T(u) = T(u_1 + \cdots + u_m) = T(u_1) + \cdots + T(u_m)$$

But since each $U_i$ is invariant under $T$ and $u_i$ is in $U_i$ ($i = 1, \cdots, m$), we have that $T(u_i)$ is in $U_i$ ($i = 1, \cdots, m$).

And hence $T(u) = v_1 + \cdots + v_m$, where $v_i = T(u_i) \in U_i$ ($i = 1, \cdots, m$), and hence by definition of $U_1 + \cdots + U_m$, we get that $T(u)$ is in $U$.

Hence $U$ is invariant under $T$. □

(3) Give an example of an operator whose matrix with respect to some basis contains only nonzero numbers on the diagonal, but the operator is not invertible.

Let $V = \mathbb{R}^2$ and define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by:

$$T(x, y) = (x + y, x + y)$$

First, let's find the matrix $A$ of $T$ with respect to the standard basis $\{(1, 0), (0, 1)\}$ of $\mathbb{R}^2$.

$$T(1, 0) = (1, 1) = \mathbf{1}(1, 0) + \mathbf{1}(0, 1)$$

$$T(0, 1) = (1, 1) = \mathbf{1}(1, 0) + \mathbf{1}(0, 1)$$

Therefore, by definition of the matrix of a linear transformation, we get that:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Notice that $A$ indeed contains only nonzero numbers on the diagonal.

However, $T$ is not invertible, because for example, even though $(1, -1) \neq (0, 0)$, we have $T(1, -1) = (1 + (-1), 1 + (-1)) = (0, 0)$. Hence $T$ is not one-to-one, and hence $T$ is not invertible!